JOURNAL OF COMBINATORIAL THEORY, Series B 24, 301–310 (1978)

Homogeneity Conditions in Graphs

A. GARDINER

Department of Pure Mathematics, University of Birmingham, P.O. Box 363, Birmingham B15 2TT, England

Communicated by the Editors

Received July 2, 1975

Let P be a class of graphs; a graph Γ with vertex set V is locally P-homogeneous if whenever $U \subseteq V$ and the vertex subgraph $\langle U \rangle$ lies in P, then each automorphism of $\langle U \rangle$ extends to an automorphism of Γ. Let C be the class of connected graphs, Q the class of cones, R the class of "rakes"; we classify locally finite, locally C-homogeneous graphs, and prove that a locally finite, locally $(Q \cup R)$-homogeneous graph is either locally C-homogeneous, or is the Levi graph of the seven-point projective plane.

1. Introduction and Main Results

We study undirected, locally finite, loopless graphs Γ, with vertex set $V = V\Gamma$, edge set $\Gamma \subseteq V \times V$, and automorphism group $\operatorname{Aut} \Gamma = G$. If $U \subseteq V$, then the *vertex subgraph* $\langle U \rangle$ has vertex set U and edge set $(U \times U) \cap \Gamma$. If $\Delta \subseteq \Gamma$ is a symmetric relation on V, then the *edge subgraph* Δ has edge set Δ, its vertex set being the domain of Δ, that is all endpoints of edges in Δ.

We have a natural metric ∂ on the set V. Set

$$\Gamma_i := \{(u, v) \in V \times V : \partial(u, v) = i\}, \qquad 0 \leqslant i,$$

and for $u \in V$

$$\Gamma_i(u) := \{v \in V : \partial(u, v) = i\}.$$

We usually write $\Gamma(u) := \Gamma_1(u)$. d denotes the *diameter* of Γ, that is $d := \sup\{i : \Gamma_i \neq \varnothing\}$. The *girth* of Γ is the length of the shortest circuit in Γ, and is denoted by g.

We write K_r for the complete graph on r vertices, $K_{k,k}$ for the complete bipartite graph of valency k, $K_{k;r}$ for the complete t-partite graph with blocks of size r, C_n for the circuit of length n, O_3 for Petersen's graph, Q_v for the k-dimensional cube, T_k for the (infinite) regular tree of valency $k \geqslant 2$.

If Γ is a graph and t is any cardinal number, then $t \cdot \Gamma$ denotes the disjoint

0095-8956/78/0243–0301$02.00/0

union of t copies of Γ, and $L(\Gamma)$ denotes the line graph of Γ. If Γ, Δ are graphs with $V\Gamma \cap V\Delta = \varnothing$, then $\Gamma \cup \Delta$ denotes the graph on the vertex set $V\Gamma \cup V\Delta$, with edge set $\Gamma \cup \Delta$. $\square_k$ is the graph which results when one identifies antipodal vertices of Q_k; thus $\square_3 \cong K_4$, $\square_4 \cong K_{4,4}$. $(2 \cdot K_{t+1})_{t-1}$ is the unique bipartite two-fold antipodal covering of K_{t+1} [6], more commonly constructed by deleting the edges of a one-factor from $K_{t+1,t+1}$.

A graph is *homogeneous* if whenever $U_1, U_2 \subseteq V$ are such that the induced subgraphs $\langle U_1 \rangle$, $\langle U_2 \rangle$ are isomorphic, then every isomorphism from $\langle U_1 \rangle$ to $\langle U_2 \rangle$ extends to an automorphism of Γ.

Infinite homogeneous graphs (relational structures) were introduced by Fraissé [3] and were studied further in [7–11]. Finite homogeneous graphs were studied by Sheehan [13].

It is elementary to show that for any homogeneous graph Γ, and any vertex $u \in V$, $\langle \Gamma(u) \rangle$ is also homogeneous. In [4] we classified (locally finite) homogeneous graphs.

THEOREM 1. *A locally finite homogeneous graph is isomorphic to one of the following*: $t \cdot K_r$ $(t, r \geqslant 1)$, $K_{t;r}$ $(t, r \geqslant 2)$, C_5, $L(K_{3,3})$.

The restriction to locally finite graphs is vital here, since Rado's remarkable universal graph [12] is also homogeneous [7].

A graph Γ is *locally homogeneous* if whenever $U \subseteq V$, each automorphism of $\langle U \rangle$ extends to an automorphism of Γ. Clearly Γ is locally homogeneous if and only if the complement Γ^c of Γ is locally homogeneous. Moreover every homogeneous graph is locally homogeneous.

By adapting the arguments of [4] slightly, one can in fact classify locally homogeneous graphs [5]; one obtains only homogeneous graphs.

THEOREM 2. *The following conditions on a locally finite graph Γ are equivalent*: (i) *Γ is homogeneous*, (ii) *Γ is locally homogeneous*, (iii) *Γ is isomorphic to one of* $t \cdot K_r$ $(t, r \geqslant 1)$, $K_{t;r}$ $(t, r \geqslant 2)$, C_5, $L(K_{3,3})$.

Theorems 1 and 2 may also be viewed as corollaries of Theorem 3.

A graph Γ is *locally C-homogeneous* if whenever $U \subseteq V$ and $\langle U \rangle$ is connected, each automorphism of $\langle U \rangle$ extends to an automorphism of Γ. A locally homogeneous graph is locally C-homogeneous. The complement of a locally C-homogeneous graph is not necessarily locally C-homogeneous; however, if Γ is locally C-homogeneous, then, for each $u \in V$, $\langle \Gamma(u) \rangle$ is locally C-homogeneous. We prove

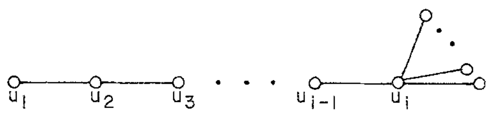
THEOREM 3. (a) *A connected locally finite graph is locally C-homogeneous if and only if it is isomorphic to one of the following*:

K_r $(r \geqslant 1)$, $K_{t;r}$ $(t, r \geqslant 2)$, C_n $(n \geqslant 5)$, $L(K_{s,s})$ $(s \geqslant 3)$, O_3, $(2 \cdot K_{t+1})_{t-1}$ $(t \geqslant 3)$, T_t $(t \geqslant 2)$, $\square_5$, $L(T_t)$ $(t \geqslant 3)$.

(b) *A disconnected locally finite graph is locally C-homogeneous if and only if each of its connected componets is locally C-homogeneous.*

The assumption that Γ is locally C-homogeneous is unacceptably restrictive. The methods used for the proof of Theorem 3 allow one to weaken this condition in various ways, none of which is very satisfactory; however, we give one example.

A graph Γ is a *cone* if $V = \{u\} \cup \Gamma(u)$, for some $u \in V$; Γ is a *rake* if it is a tree of the form



for some $i - 1$.

A graph is *locally cone homogeneous* (resp. *locally tree homogeneous, locally rake homogeneous*) if whenever $U \subseteq V$ and $\langle U \rangle$ is a cone (resp. a tree, a rake) then each automorphism of $\langle U \rangle$ extends to an automorphism of Γ.

If Γ is a locally cone homogeneous graph, then for each $u \in V$, $\langle \Gamma(u) \rangle$ is locally homogeneous and so is known from Theorem 2; however it is not clear how to classify all such graphs Γ. But if we restrict attention to graphs which are also locally rake homogeneous, then our proof yields the following result.

THEOREM 4. *A connected locally finite graph which is locally cone homogeneous and locally rake homogeneous is either locally C-homogeneous, or the Levi graph of the seven-point projective plane.*

We end this section with some basic facts about distance-transitive and distance-regular graphs.

A graph Γ is *distance transitive* if for each $i \geqslant 0$, G acts transitively on the set Γ_i. A graph Γ is *distance regular* if for each i, $0 \leqslant i \leqslant d$, and for any $(u, v) \in \Gamma_i$

$$|\Gamma(v) \cap \Gamma_{i-1}(u)| = c_i, \qquad |\Gamma(v) \cap \Gamma_i(u)| = a_i, \qquad |\Gamma(v) \cap \Gamma_{i+1}(u)| = b_i$$

depend only on i (c_0 and b_d being undefined). A distance-transitive graph is necessarily distance regular. A distance-regular graph Γ has *intersection array*

$$\iota(\Gamma) := \{b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d\},$$

which includes the a_i's implicitly since $c_i + a_i + b_i = b_0$, $1 \leqslant i < d$, and $c_d + a_d = b_0$; (a distance-regular graph Γ is regular of valency b_0). A distance-regular graph Γ is *antipodal* if $\{\{u\} \cup \Gamma_d(u): u \in V\}$ is a partition of V, (that is, $\Gamma_0 \cup \Gamma_d$ is an equivalence relation on V). With each antipodal distance-regular graph we can associate an antipodal quotient graph $\Gamma/(\Gamma_0 \cup \Gamma_d)$, whose vertices are the equivalence classes of $\Gamma_0 \cup \Gamma_d$ [6]. $\square_5$ is the antipodal quotient of Q_5; the vertices of $\square_5$ may be labeled with the 16 bipartitions of $\{1, 2, 3, 4, 5\}$, two partitions $\{A, B\}$, $\{A', B'\}$ corresponding to adjacent vertices whenever the symmetric differences $A\Delta A'$, $B\Delta B'$ each consists of one (and the same) element.

The basic facts about distance-regular graphs may be found in [2]; in particular we use the results of Hoffman–Singleton, Dammerell, Bannai–Ito on (k, g)-graphs, which are conveniently presented in [2, Chap. 23].

If $u \in V$, then G_U and $G_{[U]}$ denote, respectively, the pointwise and setwise stabilizers of U in G; if $U = \{u_1, u_2, \dots, u_i\}$, then we simply write $G_{u_1 u_2 \cdots u}$ and $G_{[u_1, u_2, \dots, u_i]}$, respectively.

S_t denotes the symmetric group on t symbols; D_{2n} denotes the dihedral group of order $2n$.

2. Proof of Main Results

Each proof is by induction. If Γ is locally C-homogeneous, then $\langle \Gamma(u) \rangle$ is also locally C-homogeneous, for each $u \in V$. Thus to prove Theorem 3 we show essentially (i) that if Γ is locally C-homogeneous, then $\langle \Gamma(u) \rangle$ occurs in the list of Theorem 3, and (ii) that if $\langle \Gamma(u) \rangle$ occurs in the list of Theorem 3, then so does Γ.

Lemma 1. *A disconnected graph is locally C-homogeneous (locally cone-homogeneous, etc.) if and only if each of its connected components is locally C-homogeneous (locally cone homogeneous, etc.).*

Lemma 2. *If Γ is locally cone homogeneous, then for each $u \in V$ $\langle \Gamma(u) \rangle$ is locally homogeneous.*

Proof. Let $u \in V$, $U \subsetneq \Gamma(u)$; each automorphism of $\langle U \rangle$ corresponds to a unique automorphism of the cone $\langle \{u\} \cup U \rangle$ fixing u (and so leaving $\Gamma(u)$ invariant), and each automorphism of $\langle \{u\} \cup U \rangle$ extends to an automorphism of Γ. Hence each automorphism of $\langle U \rangle$ extends to an automorphism of $\langle \Gamma(u) \rangle$.

Lemma 3. *If Γ is a connected locally cone homogeneous graph and $G =$ Aut Γ, then G acts transitively on V, and for each $u \in V$ G_u acts transitively on $\Gamma(u)$. If $\langle \Gamma(u) \rangle$ is disconnected, then it is the disjoint union of isomorphic complete graphs. If $\langle \Gamma(u) \rangle$ is connected, then it has diameter at most two*

Proof. If $u \in V$, then G contains elements interchanging u with each of its neighbors (K_2 is a cone); the G-orbit containing u thus contains $\Gamma(u)$. Hence each G-orbit on V is a union of connected components of Γ. Similarly each G_u-orbit on $\Gamma(u)$ is a union of connected components of $\langle\Gamma(u)\rangle$. If v_1, $v_2 \in \Gamma(u)$ and $v_2 \in \Gamma_2(v_1)$, then $G_{[u,v_1,v_2]}$ contains an element fixing u and interchanging v_1 and v_2; hence G_u acts transitively on $\Gamma(u)$. If $v_3 \in \Gamma(u) \cap \Gamma_2(v_1)$, then $G_{[u,v_1,v_2,v_3]}$ contains an element fixing u and v_1 and interchanging v_2 and v_3; thus if $\langle\Gamma(u)\rangle$ is disconnected, each connected component has diameter one and so is a complete graph. If $\langle\Gamma(u)\rangle$ is connected with diameter greater than two, then choose $v_1, v_2 \in \Gamma(u)$ with v_2 at distance three from v_1 in $\langle\Gamma(u)\rangle$; choose $v_3 \in \Gamma(v_2) \cap \Gamma(u)$ at distance two from v_1 in $\langle\Gamma(u)\rangle$. Then no element of $G_{[u,v_1,v_2,v_3]}$ can interchange v_2 and v_3. Hence $\langle\Gamma(u)\rangle$ has diameter at most two.

To classify locally C-homogeneous graphs we look for the smallest class of locally C-homogeneous graphs which contains all the graphs $t \cdot K_r$ ($r \geqslant 1$) and which is closed with respect to "extension" in the sense that if $\langle\Gamma(u)\rangle$ is in the class for some $u \in V$, then Γ is also in the class. By Lemma 1 we may restrict attention to connected graphs.

LEMMA 4. *If Γ is a connected graph such that for each $u \in V$ $\langle\Gamma(u)\rangle \cong 2 \cdot K_1$, then either $\Gamma \cong T_2$, or $\Gamma \cong C_n$ for some $n \geqslant 4$.*

LEMMA 5. *If Γ is a connected graph such that for each $u \in V$ $\langle\Gamma(u)\rangle \cong t \cdot K_1$, $t \geqslant 3$, then one of the following holds*:

(i) $\Gamma \cong K_{t,t}$, (ii) $\Gamma \cong (2 \cdot K_{t+1})_{t-1}$ *with* $\iota(\Gamma) = \{t, t-1, 1 : 1, t-1, t\}$, (iii) $\Gamma \cong O_3$, (iv) $\Gamma \cong \square_5$, (v) $\Gamma \cong T_t$, (vi) *Γ is the Levi graph of the seven-point projective plane*, (vii) *Γ is not locally rake homogeneous.*

Proof. Assume Γ is locally rake homogeneous. If Γ is acyclic, then Γ is an infinite, regular, t-valent tree. Suppose next that Γ has girth $g \geqslant 5$ with $g = 2i$ or $2i + 1$. Then if $u_0 := u$, $u_j \in \Gamma_j(u) \cap \Gamma(u_{j-1})$, $0 \leqslant j \leqslant d$, then $\langle\{u_0, u_1, ..., u_j\} \cup \Gamma(u_j)\rangle$ is a rake, $0 \leqslant j \leqslant d$, so $G_{u_0u_1 \cdots u_j}$ acts transitively on $\Gamma(u_j) - \{u_{j-1}\}$, $0 \leqslant j \leqslant d$, whence $i = d$ and Γ is a (t, g)-graph. If g is odd, then $\Gamma \cong O_3$ (the putative graph of valency $t = 57$ and diameter $d = 2$ is not distance transitive [1], and if Γ is the Hoffman–Singleton graph then $G_u \cong S_7$ so, for any $u_{-1} \in \Gamma(u_0) - \{u_1\}$ $G_{u_{-1}u_0u_1}$ cannot induce the full automorphism group S_6 on the rake $\langle\{u_{-1}, u_0, u_1\} \cup \Gamma(u_1)\rangle$).

Suppose g is even. If $g \geqslant 8$ pick $u_{d+1} \in \Gamma(u_d) - \{u_{d-1}\}$, $u_{d+2} \in \Gamma(u_{d+1}) \cap \Gamma_{d-2}(u)$, $v \in \Gamma(u_{d+1}) - \{u_d, u_{d+2}\}$; then no element of $G_{u_0u_1 \cdots u_{d+1}}$ can interchange u_{d+2} and v. Hence $g = 6$ and Γ is the Levi graph of a projective plane; since G_u acts transitively on $\Gamma_2(u)$ the collineation group acts doubly transitively on the points of the plane, so the plane is Desarguesian. But then $G = \operatorname{Aut}\Gamma$ is the extension of $P\Gamma L(3, t-1)$ by the "inverse-transpose" automorphism; now Aut Γ acts transitively on paths of length four in Γ, but $G_{u_0u_1u_2u_3}$ only induces the full symmetric group on $\Gamma(u_3) - \{u_2\}$ when $t = 3$.

Thus we may assume that Γ has girth $g = 4$. Now G_u acts transitively on $\Gamma_2(u)$, so for $w \in \Gamma_2(u) | \Gamma(w) \cap \Gamma(u)| = c_2$ does not depend on the choice of w. Moreover G_u induces the full symmetric group S_t on $\Gamma(u)$, so each c_2-subset of $\Gamma(u)$ corresponds to some vertex of $\Gamma_2(u)$. Hence

$$|\Gamma_2(u)| = \frac{t(t-1)}{c_2} \geqslant \binom{t}{c_2} \tag{2.1}$$

so $c_2 = t$, $t - 1$, or 2. If $c_2 = t$, then $\Gamma \cong K_{t,t}$. If $c_2 = t - 1$, then $|\Gamma_2(u)| = t$ and for each $w \in \Gamma_2(u)$, $\langle \Gamma(w) \rangle \cong t \cdot K_1$, so $|\Gamma_3(u)| = 1$ $\Gamma \cong (2 \cdot K_{t+1})_{t-1}$. Thus we may assume $c_2 = 2$, $t \geqslant 4$, so we have equality in (2.1) and each 2-subset of $\Gamma(u)$ corresponds to a unique vertex of $\Gamma_2(u)$. Let $w \in \Gamma_2(u)$ and suppose that we can choose $x \in \Gamma_2(u) \cap \Gamma(w)$. Then G_{uvw} acts transitively on $\Gamma(w) - \Gamma(u)$, so $\Gamma(w) - \Gamma(u) \subseteq \Gamma_2(u)$ and $d = 2$. Now $G_{\{u\} \cup (\Gamma(u) \cap \Gamma(w))}$ fixes w and induces the full symmetric group S_{t-2} on $\Gamma(u) - \Gamma(w)$, so each 2-subset of $\Gamma(u) - \Gamma(w)$ corresponds to a vertex of $\Gamma(w) \cap \Gamma_2(u)$. Hence $\binom{t-2}{2} \leqslant t - 2$, so $t = 5$, $|V| = 16$, $\langle \Gamma_2(u) \rangle \cong O_3$, and $\Gamma \cong \square_5$. Thus we may assume $\Gamma(w) \cap \Gamma_2(u) = \varnothing$. Choose $x \in \Gamma(w) \cap \Gamma_3(u)$. Then G_{uvwx} acts transitively on $(\Gamma(x) \cap \Gamma_2(u)) - \Gamma(v)$. But G_{uvwx} fixes $\Gamma(u) \cap \Gamma(w) =: \{v, v'\}$, $\Gamma(v') \cap \Gamma(x) =: \{w, w'\}$, and $\Gamma(x) \cap \Gamma(v') =: \{w, w''\}$. Hence $\Gamma(x) \cap \Gamma_2(u) = \{w, w', w''\}$. Since $t \geqslant 4$ we may choose $y \in \Gamma(x) - \Gamma_2(u)$; but then no element of G_{uvwx} can interchange w'' and y so we have a contradiction.

Observe that the Levi graph of the seven-point projective plane is locally cone homogeneous, but not locally tree homogeneous since G_u does not induce the full automorphism group $S_2 \wr S_3$ on $\Gamma_2(u)$.

LEMMA 6. *If Γ is a connected graph such that for each $u \in V$ $\langle \Gamma(u) \rangle \cong t \cdot K_r$ $(t, r \geqslant 2)$, then $\Gamma \cong L(T_{r+1})$, $\Gamma \cong L(K_{r+1,r+1})$, or Γ is not both locally cone homogeneous and locally rake homogeneous.*

Proof. Let $\Gamma(u) = U_1 \cup U_2 \cup \cdots \cup U_t$ be the decomposition of $\Gamma(u)$ into connected components. For each $v \in V$, G_v induces the full group of automorphisms on vertex subgraphs of $\langle \Gamma(v) \rangle$ isomorphic to $t \cdot K_1$ or $(t-1) \cdot K_1 \cup K_2$. Hence G_u acts transitively on the set of all vertex subgraphs of $\langle \Gamma(u) \rangle$ isomorphic to $j \cdot K_1$, for each j, $1 \leqslant j \leqslant t$. Moreover if $v \in \Gamma(u)$, then G_{uv} acts transitively on $\Gamma(v) \cap \Gamma_2(u)$, so G_u acts transitively on $\Gamma_2(u)$. Thus for $w \in \Gamma_2(u)$, $|\Gamma(w) \cap \Gamma(u)| = c_2$ is independent of the choice of w. But G_u acts transitively on the set of vertex subgraphs $c_2 \cdot K_1$ of $\langle \Gamma(u) \rangle$, so each such subgraph corresponds to a vertex of $\Gamma_2(u)$. Hence

$$\binom{t}{c_2} \cdot r^{c_2} \leqslant |\Gamma_2(u)| = \frac{tr(tr-r)}{c_2} \tag{2.2}$$

so $c_2 \leqslant 2$.

Suppose $c_2 = 2$. Then we have equality in (2.2), so each subgraph $2 \cdot K_1$ of $\langle \Gamma(u) \rangle$ corresponds to a unique vertex of $\Gamma_2(u)$. Choose $w \in \Gamma_2(u)$ and let $\Gamma(u) \cap \Gamma(w) = \{v, v'\}$. Now $\langle \Gamma(w) \cap \Gamma(v') \cap \Gamma_2(u) \rangle \cong K_{r-1}$ so choose $x \in \Gamma_2(u) \cap \Gamma(v') \cap \Gamma(w)$. If $X \subseteq \Gamma(w) - (\Gamma(u) \cap \Gamma(v))$ is such that $x \in X$ and $\langle X \rangle \cong j \cdot K_1$, for some $j \geqslant 1$, then $\langle \{u, v, w\} \cup X \rangle$ is a rake, so $X \subseteq \Gamma_2(u)$. Hence $d = 2$. If $t \geqslant 3$, then G_{uvw} acts transitively on $\langle \Gamma(w) - (\Gamma(u) \cap \Gamma(v)) \rangle = (t-2) \cdot K_r \cup K_{r-1}$, which is evidently impossible. Hence $t = 2$, so $\Gamma = L(\Delta)$ is a line graph; Δ is bipartite of diameter two, girth four, valency $r+1$, so $\Gamma = L(K_{r+1,r+1})$.

Suppose $c_2 = 1$. We show that $t = 2$. Suppose on the contrary that $t \geqslant 3$. Let $u =: u_0$, and $u_{j+1} \in \Gamma(u_j) \cap \Gamma_{j+1}(u)$, $0 \leqslant j < d$. Now G_u acts transitively on $\Gamma(u)$ and $\Gamma_2(u)$. We show that Γ is distance transitive. Suppose G_u acts transitively on $\Gamma_j(u)$. Choose $z \in \Gamma_j(u)$; then $|\Gamma(z) \cap \Gamma_{j+1}(u)| = b_j$ is independent of the choice of z. In particular if $j < d$, then $b_j \neq 0$. For any $X \subseteq \Gamma(u_j) - (\Gamma(u_{j-1}) \cup \{u_{j-1}\})$ such that $\langle X \rangle \cong (t-1) \cdot K_1$, $G_{u_0 u_1 \cdots u_j}$ acts transitively on X. It follows that if $j < d$, then $b_j = tr - r$ and $G_{u_0 u_1 \cdots u_j}$ acts transitively on $\Gamma(u_j) \cap \Gamma_{j+1}(u)$; hence G_u acts transitively on $\Gamma_{j+1}(u)$ and Γ is distance transitive. Choose any nonadjacent vertices $u_{d+1}, u'_{d+1} \in \Gamma(u_d) - (\Gamma(u_{d-1}) \cup \{u_{d-1}\})$. Let v be the unique vertex (a) in $\Gamma_{d-1}(u_{d+1}) \cap \Gamma(u)$ if $u_{d+1} \in \Gamma_d(u)$, (b) in $\Gamma_{d-2}(u_{d+1}) \cap \Gamma(u)$ if $u_{d+1} \in \Gamma_{d-1}(u)$; choose $v' \in \Gamma(v) \cap \Gamma(u)$. Then $\langle v', u_0, u_1, \ldots, u_{d+1}, u'_{d+1} \rangle$ is a rake, but no element of $G_{v' u_0 u_1 \cdots u_d}$ can interchange u_{d+1} and u'_{d+1}. Thus $t = 2$ so $\Gamma = L(\Delta)$ is a line graph, and Δ has girth $\gamma \geqslant 5$, or Δ is acyclic so $\Delta \cong T_{r+1}$. Assume Δ has finite girth.

Now $c_2 = 1$, so the bipartition $\Gamma(u) = U_1 \cup U_2$ induces a bipartition $\Gamma_2(u) = V_1 \cup V_2$, such that if $v_i \in V_i$, then $\Gamma(v_i) \cap \Gamma(u) \subseteq U_i$, $i = 1, 2$; further $|\Gamma(v_i) \cap V_i| = r - 1$. Since $c_2 = 1$ and $\langle \Gamma(v_i) \rangle \cong 2 \cdot K_r$, we have either $\Gamma(v_i) \cap \Gamma_2(u) \subseteq V_i$ and $\gamma \geqslant 6$, or $|\Gamma(v_i) \cap \Gamma_2(u)| = r$ and $\gamma = 5$. Choose $x_1 \in U_1$, $x_2 \in U_2$; if $v_1 \in V_1 \cap \Gamma(x_1)$, $v_2 \in (V_2 \cap \Gamma(x_2)) - \Gamma(v_1)$, then $G_{[u, x_1, x_2, v_1, v_2]}$ contains an element interchanging x_1 and x_2, v_1 and v_2. Since $r \geqslant 2$, it follows that $G_{x_2 u x_1}$ acts transitively on $\Gamma(x_1) \cap \Gamma_2(u)$; hence G acts transitively on paths of length four in Δ. Similarly if Δ has girth $\gamma > 2i$, then G acts transitively on paths of length $2i$ in Δ. But this forces either $r = 1$, $\Delta \cong C_n$ for some n, or $\gamma \leqslant 4$. Thus the lemma is proved.

$L(K_{r+1,r+1})$ *is* locally C-homogeneous, since connected vertex subgraphs of $L(K_{r+1,r+1})$ correspond in a one-to-one fashion to connected edge subgraphs of $K_{r+1,r+1}$, and if F is a subset of edges of $K_{r+1,r+1}$, then each automorphism of F extends to an automorphism of $K_{r+1,r+1}$.

LEMMA 7. *If Γ is a connected graph such that for each $u \in V$ $\langle \Gamma(u) \rangle \cong K_{t;r} (t, r \geqslant 2)$, then $\Gamma \cong K_{t+1;r}$.*

Proof. Choose $v \in \Gamma(u)$. *Then* $\Gamma(v) \cap \Gamma_2(u) = \{u_2, \ldots, u_r\}$, say. Now $\langle \Gamma(v) \rangle \cong K_{t;r}$ so for each $w \in \Gamma(v) \cap \Gamma(u)$ we have $\{u_2, \ldots, u_r\} \subseteq \Gamma(w)$. Since $\langle \Gamma(u) \rangle$ is connected we obtain $\Gamma(u_i) = \Gamma(u)$, $2 \leqslant i \leqslant r$, so $\Gamma \cong K_{t+1;r}$.

LEMMA 8. *If Γ is a connected graph such that for each $u \in V$ $\langle \Gamma(u) \rangle \cong K_t$ $(= K_{t;1})$, then $\Gamma \cong K_{t+1}$.*

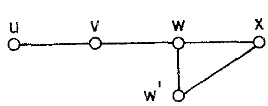
LEMMA 9. *If Γ is a connected graph such that for each $u \in V$ $\langle \Gamma(u) \rangle \cong C_n$, then one of the following holds:* (i) $n = 3$, $\Gamma \cong K_4$, (ii) $n = 4$, $\Gamma \cong K_{3;2}$, (iii) $n = 5$, $\Gamma \cong (2.K_1)_2$ *(the icosahedron)*, (iv) $n \geqslant 6$ *and Γ is not locally cone homogeneous.*

Proof. (i) and (ii) follow from Lemmas 8 and 7, respectively. (iv) follows from Lemma 2 since, for $n \geqslant 6$, C_n is not locally homogeneous. Assume $n = 5$ and $\langle \Gamma(u) \rangle \cong C_5$; for each $v \in \Gamma(u)$ we have $\langle \Gamma(v) \rangle \cong C_5$, so $\langle \Gamma_2(u) \rangle \cong C_5$, $|\Gamma_3(u)| = 1$, $\Gamma \cong (2 \cdot K_6)_2$.

LEMMA 10. *If Γ is a connected graph such that for each $u \in V$ $\langle \Gamma(u) \rangle \cong L(K_{3,3})$, then Γ is the graph $(2 \cdot K_{10})_4$ with $\iota(\Gamma) = \{9, 4, 1; 1, 4, 9\}$.*

Proof. For $v \in \Gamma(u)$, $\langle \Gamma(v) \cap \Gamma(u) \rangle \cong 2 \cdot K_2$. If $w \in \Gamma(v) \cap \Gamma_2(u)$, then since $\langle \Gamma(v) \rangle \cong L(K_{3,3})$ we have $\langle \Gamma(w) \cap \Gamma(u) \cap \Gamma(v) \rangle \cong 2 \cdot K_1$. Moreover for each $w' \in \Gamma_2(u)$, $\langle \Gamma(w') \cap \Gamma(u) \rangle \cong C_4$ so $|\Gamma_2(u)| = 9 \cdot 4/4 = 9$. But $\langle \Gamma(w) \rangle \cong L(K_{3,3})$, $\langle \Gamma(w) \cap \Gamma(v) \rangle \cong 2 \cdot K_2$, etc.. Thus we see that $\Gamma(w) \cap \Gamma_2(u) \supseteq W$, where $\langle W \rangle \cong 2 \cdot K_2$, and $\Gamma(w) - \Gamma(u) = W \cup \{u'\}$ with $W \subseteq \Gamma(u')$. In particular $|\Gamma_3(u)| \leqslant 1$, $|\Gamma(w) \cap \Gamma_2(u)| \geqslant 4$. If $w' \in W$, then $\Gamma(w') - \Gamma(u) \supseteq W'$ with $\langle W' \rangle \cong 2 \cdot K_2$ and $\langle \Gamma(w') - \Gamma(u) \rangle \cong \langle \Gamma(w) - \Gamma(u) \rangle$. Hence $\Gamma(w') - \Gamma(u) = W' \cup \{u'\}$. Continuing in this fashion we see that $\Gamma_2(u) = \Gamma(u')$, $\{u'\} = \Gamma_3(u)$, $\langle \Gamma_2(u) \rangle \cong L(K_{3,3})$, $\Gamma \cong (2 \cdot K_{10})_4$; (it is elementary to show that there is only one twofold antipodal cover of K_{10} with the given intersection array).

The graph $(2 \cdot K_{10})_4$ is distance transitive. This is due to the fact that Aut K_{10} contains a subgroup isomorphic to S_6 which is doubly transitive on the 10 vertices; $\bar{S}_6$ has a nonsplit central extension $\bar{\bar{S}}_6$ with center of order 2 which is realized as a section of $P\Gamma L(2, 9)$; $\bar{\bar{S}}_6$ has a subgroup isomorphic to $S_3 \wr S_2$ of index 20 and $\Gamma \cong (2 \cdot K_{10})_4$ is a generalized Cayley graph on the cosets of such a subgroup. Thus a vertex stabilizer G_u induces the full automorphism group on $\langle \Gamma(u) \rangle \cong L(K_{3,3})$, so the graph is locally cone homogeneous. Similarly the graph $(2 \cdot K_6)_2$ is locally cone homogeneous. However, neither graph is locally rake homogeneous, since each graph contains a path of length four on which only the identity automorphism is induced; further both $(2 \cdot K_6)_2$ and $(2 \cdot K_{10})_4$ contain vertex subgraphs of the form



with $w' \in \Gamma_2(u)$, $x \in \Gamma_3(u)$, so no element of G_{uvw} can interchange w' and x. Thus neither graph is locally C-homogeneous.

LEMMA 11. *None of the graphs* $T_t\,(t \geqslant 2)$, $L(T_t)\,(t \geqslant 2)$, O_3, $\square_5$, $(2 \cdot K_{t+1})_{t-1}\,(t \geqslant 2)$, $C_n\,(n \geqslant 6)$, $L(K_{t,t})\,(t \geqslant 4)$, *is locally homogeneous.*

Proof. This follows from Theorem 2, but we give a separate proof so that Theorem 2 may be deduced as a corollary of Theorem 3.

(a) If $\Gamma \cong T_t\,(t \geqslant 2)$, $L(T_t)\,(t \geqslant 2)$, $(2 \cdot K_{t+1})_{t-1}\,(t \geqslant 2)$, or $C_n\,(n \geqslant 6)$, we may pick $u \in V$, $v \in \Gamma_2(u)$, $w \in \Gamma(v) \cap \Gamma_3(u)$, whence no element of $G_{[u,v,w]}$ can interchange v and w.

(b) O_3 contains a vertex subgraph isomorphic to $3 \cdot K_2$ on which Aut $O_3 \cong S_5$ does not induce the full automorphism group $S_2 \wr S_3$.

(c) If $\Gamma \cong \square_5{}^c$ and $u \in V$, then $\langle \Gamma(u) \rangle \cong O_3{}^c$; but O_3 is not locally homogeneous by (b), so neither is $O_3{}^c$. Hence $\square_5{}^c$ is not locally homogeneous, so neither is $\square_5$.

(d) $L(K_{t,t})\,(t \geqslant 4)$ contains a vertex subgraph isomorphic to $2 \cdot C_4$ on which Aut $L(K_{t,t}) \cong S_t \wr S_2$ does not induce the full automorphism group $D_8 \wr S_2$.

Proof of Theorem 3 (resp. Theorem 4). If Γ is locally C-homogeneous (resp. locally cone homogeneous *and* locally rake homogeneous), then $\langle \Gamma(u) \rangle$ is locally homogeneous and the sequence $\Gamma =: \Gamma^1$, $\langle \Gamma^1(u_1) \rangle =: \Gamma^2$, $\langle \Gamma^2(u_2) \rangle =: \Gamma^3, \ldots$ must terminate with a complete graph K_r. Thus the class of graphs required is the smallest class of locally C-homogeneous graphs (resp. locally cone homogeneous and locally rake homogeneous graphs) which contains the graphs K_r and which is closed with respect to "extension." This class must then contain all graphs listed in Theorem 3 (resp. Theorem 4), and Lemmas 4 through 11 imply that this list is complete.

REFERENCES

1. M. ASCHBACHER, The nonexistence of rank three permutation groups of degree 3250 and subdegree 57, *J. Algebra* **19** (1971), 538–540.
2. N. L. BIGGS, "Algebraic Graph Theory," Cambridge Tracts in Mathematics 67, Cambridge Univ. Press, London/New York, 1974.
3. R. FRAISSÉ, Sur l'extension aux relations de quelques propriétés des ordres, *Ann. Sci. École Norm. Sup.* **71** (1954), 361–388.
4. A. GARDINER, Homogeneous graphs, *J. Combinatorial Theory Ser. B* **20** (1976), 94–102.
5. A. GARDINER, Homogeneous graphs and stability, *J. Austral. Math. Soc.*, to appear.
6. A. GARDINER, Antipodal covering graphs, *J. Combinatorial Theory Ser. B* **16** (1974), 255–273.
7. C. W. HENSON, A family of countable homogeneous graphs, *Pacific J. Math.* **38** (1971), 69–83.
8. C. W. HENSON, Countable homogeneous relational structures and categorical theories, *J. Symbolic Logic* **37** (1972), 494–500.
9. B. JOHNSSON, Universal relational systems, *Math. Scand.* **4** (1956), 193–208.

10. B. JONSSON, Homogeneous universal relational systems, *Math. Scand.* **8** (1960), 137–142.
11. M. MORLEY AND R. VAUGHT, Homogeneous universal models, *Math. Scand.* **11** (1962), 37–57.
12. R. RADO, Universal graphs and Universal functions, *Acta Arith.* **9** (1964), 331–340.
13. J. SHEEHAN, Smoothly embeddable subgraphs, *J. London Math. Soc.* **9** (1974), 212–218.